

# COMMUTING CATEGORIES FOR BLOCKS AND FUSION SYSTEMS

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**ABSTRACT.** We extend the notion of a commuting poset for a finite group to  $p$ -blocks and fusion systems, and we generalize a result, due originally to Alperin and proved independently by Aschbacher and Segev, to commuting graphs of blocks, with a very short proof based on the  $G$ -equivariant version, due to Thévenaz and Webb, of a result of Quillen.

Let  $k$  be a field of prime characteristic  $p$ . A *block of a finite group  $G$*  is a primitive idempotent  $b$  in  $Z(kG)$ . A  *$b$ -Brauer pair* is a pair  $(Q, e)$  consisting of a  $p$ -subgroup  $Q$  of  $G$  and a block  $e$  of  $C_G(Q)$  satisfying  $\text{Br}_Q(b)e \neq 0$ , where  $\text{Br}_Q : (kG)^Q \rightarrow kC_G(Q)$  is the Brauer homomorphism; the set of  $b$ -Brauer pairs is a  $G$ -poset with respect to the conjugation action of  $G$  (see [10] for more details and background material on block theory). We denote by  $\mathcal{A}(b)$  the  $G$ -poset containing all  $b$ -Brauer pairs  $(Q, e)$  such that  $Q$  is nontrivial and elementary abelian.

Two subgroups  $R, R'$  of  $G$  are said to *commute* if they commute elementwise; that is, if  $[R, R'] = 1$ . For any nonempty set  $\kappa$  of pairwise commuting subgroups of  $G$  we denote by  $\Pi\kappa$  the product in  $G$  of all subgroups belonging to  $\kappa$ ; this is clearly a subgroup of  $G$ . If all elements of  $\kappa$  are  $p$ -subgroups (respectively, abelian subgroups) of  $G$ , then  $\Pi\kappa$  is a  $p$ -subgroup (respectively, abelian subgroup) of  $G$ . For any abelian subgroup  $Q$  of  $G$  we denote by  $c(Q)$  the set of subgroups of order  $p$  of  $Q$ .

**Definition 1.** Let  $G$  be a finite group and  $b$  a block of  $G$ . The *commuting poset of  $b$*  is the  $G$ -poset  $\mathcal{K}(b)$  whose elements are pairs  $(\kappa, e)$ , where  $\kappa$  is a nonempty set of pairwise commuting subgroups of order  $p$  of  $G$  and where  $e$  is a block of  $C_G(\Pi\kappa)$  such that  $(\Pi\kappa, e)$  is a  $b$ -Brauer

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pair, with partial order given by

$$(\lambda, f) \leq (\kappa, e), \text{ if } \begin{cases} \lambda \subseteq \kappa, \text{ and} \\ (\Pi\lambda, f) \leq (\Pi\kappa, e) \end{cases}$$

for  $(\kappa, e), (\lambda, f) \in \mathcal{K}(b)$ .

If  $b$  is the principal block of  $G$  then  $\mathcal{K}(b)$  is the clique complex  $\mathcal{K}_p(G)$  of the commuting graph  $\Lambda_p(G)$ , where the notation is as in [3]. For nonprincipal blocks, however,  $\mathcal{K}(b)$  need not be the clique complex of a graph (e.g., see Example 5).

Given a  $G$ -poset  $\mathfrak{X}$  we denote by  $\Delta\mathfrak{X}$  the  $G$ -simplicial complex whose set of  $n$ -simplices consists of all chains of  $n$  proper inclusions in  $X$ , where  $n \geq 0$ . For any simplicial complex  $\mathfrak{Y}$ , we denote the geometric realization of  $\mathfrak{Y}$  by  $|\mathfrak{Y}|$ . Two  $G$ -spaces  $X$  and  $Y$  are called  *$G$ -homotopically equivalent* if there are  $G$ -equivariant maps  $f : X \rightarrow Y, g : Y \rightarrow X$  and  $G$ -equivariant homotopies  $h : I \times X \rightarrow X, h' : I \times Y \rightarrow Y$  such that  $h(0, -) = \text{Id}_X, h(1, -) = f, h'(0, -) = \text{Id}_Y$ , and  $h'(1, -) = g$ , where the unit interval  $I = [0, 1]$  is viewed as a  $G$ -space with the trivial  $G$ -action. Two  $G$ -posets  $\mathfrak{X}$  and  $\mathfrak{Y}$  are called  *$G$ -homotopically equivalent* if the  $G$ -spaces  $|\Delta\mathfrak{X}|$  and  $|\Delta\mathfrak{Y}|$  are  $G$ -homotopically equivalent. By the  $G$ -equivariant version [11, (1.1)] of [9, 1.3], in order to show that  $\mathfrak{X}$  and  $\mathfrak{Y}$  are  $G$ -homotopically equivalent, it suffices to find  $G$ -equivariant functors  $\Phi : \mathfrak{X} \rightarrow \mathfrak{Y}$  and  $\Psi : \mathfrak{Y} \rightarrow \mathfrak{X}$  such that there is a natural transformation between  $\text{Id}_{\mathfrak{X}}$  and  $\Psi \circ \Phi$  (in either direction) and a natural transformation between  $\text{Id}_{\mathfrak{Y}}$  and  $\Phi \circ \Psi$ .

**Theorem 2.** *Let  $b$  be a block of a finite group  $G$ . The maps:*

$$\Phi : \begin{cases} \mathcal{A}(b) \rightarrow \mathcal{K}(b) \\ (Q, e) \mapsto (c(Q), e) \end{cases} \quad \text{and} \quad \Psi : \begin{cases} \mathcal{K}(b) \rightarrow \mathcal{A}(b) \\ (\kappa, e) \mapsto (\Pi\kappa, e) \end{cases}$$

*are inverse  $G$ -homotopy equivalences.*

*Proof.* The maps  $\Phi, \Psi$  are obviously order preserving and  $G$ -equivariant. We have  $\Psi \circ \Phi = \text{Id}_{\mathcal{A}(b)}$ . There is a natural transformation  $\text{Id}_{\mathcal{K}(b)} \rightarrow \Phi \circ \Psi$  given by  $(\kappa, e) \leq (c(\Pi\kappa), e)$ , which shows that  $\Psi$  is a  $G$ -homotopy inverse of  $\Phi$ .  $\square$

Applied to principal blocks, this theorem yields, in particular, a proof of the fact, due independently to Alperin [1, Theorem 3] and to Aschbacher and Segev [4, 9.7], that  $\mathcal{K}_p(G)$  and  $\mathcal{A}_p(G)$  have the same homotopy type (see also [3, 5.2]). The  $G$ -orbit space of  $\mathcal{K}(b)$  admits a generalization to fusion systems and, in fact, to arbitrary categories on finite  $p$ -groups (cf. [7, 2.1]).

**Definition 3.** Let  $\mathcal{F}$  be a category on a finite  $p$ -group  $P$ . The *commuting category* of  $\mathcal{F}$  is the category  $\mathcal{K}(\mathcal{F})$  whose objects are the nonempty sets of pairwise commuting subgroups of  $P$  of order  $p$ , and for objects  $\kappa, \lambda \in \mathcal{K}(\mathcal{F})$ ,

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{F})}(\kappa, \lambda) = \{\psi \in \mathrm{Hom}_{\mathcal{F}}(\Pi\kappa, \Pi\lambda) \mid \text{if } Q \in \kappa, \text{ then } \psi(Q) \in \lambda.\}$$

The composition of morphisms in  $\mathcal{K}(\mathcal{F})$  is induced by the usual composition of group homomorphisms. We denote by  $[\mathcal{K}(\mathcal{F})]$  the poset consisting of the isomorphism classes  $[\kappa]$  of objects  $\kappa$  of  $\mathcal{K}(\mathcal{F})$  with partial order given by

$$[\kappa] \leq [\lambda], \text{ if } \mathrm{Hom}_{\mathcal{K}(\mathcal{F})}(\kappa, \lambda) \neq \emptyset$$

for  $\kappa, \lambda \in \mathcal{K}(\mathcal{F})$ .

Clearly  $\mathcal{K}(\mathcal{F})$  is an *EI*-category. As a consequence of results in [2], any choice of a maximal  $b$ -Brauer pair  $(P, e)$  of a block  $b$  of a finite group  $G$  determines a category  $\mathcal{F}_{(P, e)}(G, b)$  on  $P$  that, if  $k$  is large enough, is a saturated fusion system (see e.g., [6, §3.3] for details and further references).

**Theorem 4.** *Let  $b$  be a block of a finite group  $G$ , let  $(P, e_P)$  be a maximal  $b$ -Brauer pair and let  $\mathcal{F} = \mathcal{F}_{(P, e_P)}(G, b)$ . We have an isomorphism of posets*

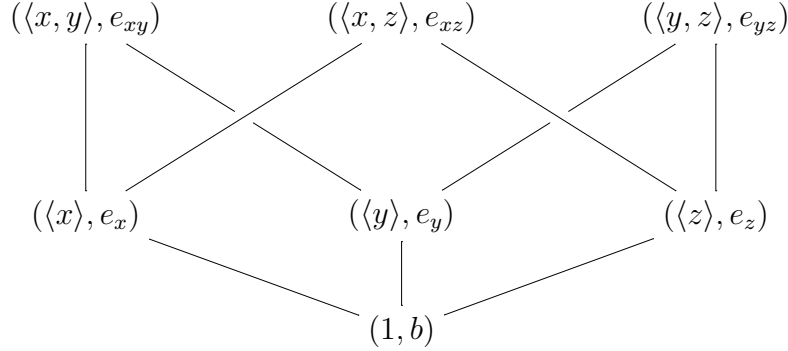
$$[\mathcal{K}(\mathcal{F})] \cong \mathcal{K}(b)/G$$

*mapping the isomorphism class of an object  $\kappa \in \mathcal{K}(\mathcal{F})$  to the  $G$ -conjugacy class of the unique Brauer pair  $(\Pi\kappa, e)$  contained in  $(P, e_P)$ .*

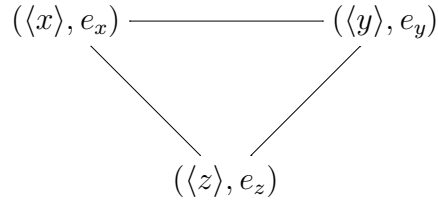
*Proof.* For  $(\kappa, e) \in \mathcal{K}(b)$ , let  $[(\kappa, e)]$  denote its  $G$ -conjugacy class. For elements  $(\kappa, e), (\lambda, f) \in \mathcal{K}(b)$ , one has  $[(\kappa, e)] = [(\lambda, f)]$  if and only if there exists  $g \in G$  such that  $\kappa^g = \lambda$  and  $e^g = f$ . Define a poset map  $\eta : \mathcal{K}(b)/G \rightarrow [\mathcal{K}(\mathcal{F})]$  by setting  $\eta([( \kappa, e)]) = [\kappa^g]$ , where  $g \in G$  such that  $(\Pi\kappa, e)^g \leq (P, e_P)$ . One verifies that this map is the inverse of the given map in the statement.  $\square$

**Example 5.** The following example was communicated to the authors by R. Kessar. Suppose  $p = 2$ . Set  $G = S_n$ , where  $n \geq 6$  is an integer such that  $kG$  has a block  $b$  with a dihedral defect group  $P \cong D_8$  of order 8. By results in [8],  $b$  is of principal type; that is, for any 2-subgroup  $Q$  of  $G$  either  $\mathrm{Br}_Q(b) = 0$  or  $\mathrm{Br}_Q(b)$  is a block of  $kC_G(Q)$ . Moreover,  $P$  may be chosen as a Sylow 2-subgroup of  $S_4$ , canonically embedded into  $G$  and such that  $P$  contains the involutions  $x = (1\ 2)$ ,  $y = (3\ 4)$ . Setting  $z = (5\ 6)$ , we have  $x, z \in P^{(3\ 5)(4\ 6)}$  and  $y, z \in P^{(1\ 5)(2\ 6)}$ . Since  $b$  is of principal type, there are unique blocks  $e_x, e_y, e_z$  of  $kC_G(x), kC_G(y), kC_G(z)$ , respectively, and unique blocks

$e_{xy}, e_{xz}, e_{yz}$  of  $kC_G(\langle x, y \rangle)$ ,  $kC_G(\langle x, z \rangle)$ ,  $kC_G(\langle y, z \rangle)$ , respectively, giving the following inclusions of  $b$ -Brauer pairs:



Suppose that  $\Gamma$  is a graph whose clique complex is  $\mathcal{K}(b)$ . The  $b$ -Brauer pairs  $(\langle x \rangle, e_x)$ ,  $(\langle y \rangle, e_y)$ , and  $(\langle z \rangle, e_z)$  are minimal in the poset  $\mathcal{K}(b)$  and are pairwise contained in a common  $b$ -Brauer pair, implying that the graph  $\Gamma$  has a clique of the form:



However, the corresponding clique is not an element of the poset  $\mathcal{K}(b)$  because the group  $\langle x, y, z \rangle$  is not contained in a defect group of  $b$ . This contradiction shows that there is no graph whose clique complex yields  $\mathcal{K}(b)$  and explains why we have refrained from defining a commuting graph of  $b$  in this way.

## REFERENCES

- [1] J. L. Alperin, *A Lie approach to finite groups*, Lecture Notes in Math. **1456**, Springer, Berlin (1990), 1–9.
- [2] J. Alperin, M. Broué, *Local methods in block theory*, Ann. Math. **110** (1979), 143–157.
- [3] M. Aschbacher, *Simple connectivity of  $p$ -group complexes*, Israel J. Math. **82** (1993), 1–43.
- [4] M. Aschbacher, Y. Segev, *The uniqueness of groups of Lyons type*, J. Amer. Math. Soc. **5** (1992), 75–98.
- [5] C. Broto, R. Levi, B. Oliver, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. **16** (2003), 779–856.

- [6] R. Kessar, *Introduction to Block Theory*, in: Group Representation Theory (eds. M. Geck, D. Testerman, J. Thévenaz), EPFL Press, Lausanne (2007), 47–77.
- [7] M. Linckelmann, *Introduction to Fusion systems*, in: Group Representation Theory (eds. M. Geck, D. Testerman, J. Thévenaz), EPFL Press, Lausanne (2007), 79–113.
- [8] L. Puig, *The Nakayama conjecture and the Brauer pairs*, Sminaire sur les groupes finis, Tome III, ii, 171189, Publ. Math. Univ. Paris VII, 25, Univ. Paris VII, Paris (1986).
- [9] D. Quillen, *Homotopy properties of the Poset of Nontrivial  $p$ -Subgroups of a Group*, Advances Math. **28** (1978), 101–128.
- [10] J. Thévenaz,  *$G$ -Algebras and Modular Representation Theory*, Oxford Science Publications, Clarendon Press, Oxford (1995).
- [11] J. Thévenaz, P. J. Webb, *Homotopy equivalence of posets with a group action*, J. Combin. Theory Ser. A **56** (1991), no. 2, 173–181.

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